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FINITE GROUPS WITH MANY NORMALIZERS

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Abstract. A group G is said to have dense normalizers if every nonempty open interval in its subgroup lattice $L(G)$ contains the normalizer of a certain subgroup of G. We find all finite groups satisfying this property. We also classify the finite groups, in which k subgroups are not normalizers for $k = 1, 2, 3, 4$.

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1. Introduction

Let $\mathcal{X}$ be a property pertaining to subgroups of a group. We say that a group G has *dense $\mathcal{X}$-subgroups* if for every pair (H, K) of subgroups of G such that $H < K$ and H is not maximal in K, there exists a $\mathcal{X}$-subgroup X of G such that $H < X < K$. Groups with dense $\mathcal{X}$-subgroups have been studied for many properties $\mathcal{X}$: being a normal subgroup (see [9]), a pronormal subgroup (see [14]), a normal-by-finite subgroup (see [4]), a nearly normal subgroup (see [5]) or a subnormal subgroup, see [6].

Our first result deals with the property of being the normalizer of a certain subgroup.

Theorem 1.1. *A finite group G has dense normalizers if and only if G is either cyclic of prime order or non-Abelian of order pq for some primes p and q.*

Next we study finite groups with many normalizers in another sense. Let G be a finite group, $L(G)$ be the subgroup lattice of G, and $N_G = \{N_G(H)\colon\ H \in L(G)\}$. Obviously, we have $|N_G| = |L(G)|$ if and only if G is trivial. Our second result describes finite groups G satisfying $|N_G| = |L(G)| - k$, where $k = 1, 2, 3$. Finite groups G such that $|N_G| = |L(G)| - 4$ are also determined modulo ZM-groups.

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Theorem 1.2. *For a finite group G, we have:*

(a) $|N_G| = |L(G)| - 1$ *if and only if* $G \cong \mathbb{Z}_p$ *with a prime* p.

(b) $|N_G| = |L(G)| - 2$ *if and only if* $G \cong \mathbb{Z}_{p^2}$ *with a prime* p *or* $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ *with primes* p, q *satisfying* $q \mid p-1$.

(c) $|N_G| = |L(G)| - 3$ *if and only if* $G \cong \mathbb{Z}_{p^3}$ *with a prime* p *or* $G \cong \mathbb{Z}_{pq}$ *with distinct primes* p, q *or* $G \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$ *with primes* p, q *satisfying* $q \mid p-1$.

(d) *If G is not a* ZM*-group, then* $|N_G| = |L(G)| - 4$ *if and only if* $G \cong \mathbb{Z}_{p^4}$ *with a prime* p *or* $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $G \cong A_4$.

We note that finite groups G with few normalizers, i.e. with small number $|N_G|$, have been studied in [8], [10], [15], [16]. Also, we remark that item (d) in Theorem 1.2 leads to the following characterization of A_4.

Corollary 1.3. *A_4 is the unique finite non-Abelian group G satisfying $|N_G| = |L(G)| - 4$ and having at least one noncyclic Sylow subgroup.*

We recall that a ZM-*group* is a finite non-Abelian group with all Sylow subgroups cyclic. By [7], such a group is of type

$$\mathrm{ZM}(m,n,r) = \langle a, b\colon\ a^m = b^n = 1,\ b^{-1}ab = a^r\rangle,$$

where the triple (m,n,r) satisfies the conditions

$$\gcd(m,n) = \gcd(m, r-1) = 1 \quad \text{and} \quad r^n \equiv 1 \pmod{m}.$$

It is clear that $|\mathrm{ZM}(m,n,r)| = mn$ and $Z(\mathrm{ZM}(m,n,r)) = \langle b^d\rangle$, where d is the multiplicative order of r modulo m, i.e.,

$$d = o_m(r) = \min\{k \in \mathbb{N}^*\colon\ r^k \equiv 1 \pmod{m}\}.$$

The subgroups of $\mathrm{ZM}(m,n,r)$ have been completely described in [1]. Set

$$L = \left\{(m_1, n_1, s) \in \mathbb{N}^3\colon\ m_1 \mid m,\ n_1 \mid n,\ s < m_1,\ m_1 \mid s\frac{r^n - 1}{r^{n_1} - 1}\right\}.$$

Then there is a bijection between L and the subgroup lattice $L(\mathrm{ZM}(m,n,r))$ of $\mathrm{ZM}(m,n,r)$, namely the function that maps a triple $(m_1, n_1, s) \in L$ into the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)} = \bigcup_{k=1}^{n/n_1} \alpha(n_1, s)^k \langle a^{m_1}\rangle = \langle a^{m_1}, \alpha(n_1, s)\rangle,$$

where $\alpha(x,y) = b^x a^y$ for all $0 \leqslant x < n$ and $0 \leqslant y < m$. We remark that $|H_{(m_1,n_1,s)}| = mn/(m_1 n_1)$ for any s satisfying $(m_1, n_1, s) \in L$.

For the proof of Theorem 1.2, we need the following well-known result on p-groups, see e.g. (4.4) of [13].

Theorem A. *The following conditions on a finite p-group G are equivalent:*
(a) *Every Abelian subgroup of G is cyclic.*
(b) *G has a unique subgroup of order p.*
(c) *G is either cyclic or a generalized quaternion 2-group.*

By a *generalized quaternion 2-group* we understand a group of order 2^n for some positive integer $n \geqslant 3$, defined by

$$Q_{2^n} = \langle a, b\colon\ a^{2^{n-2}} = b^2,\ a^{2^{n-1}} = 1,\ b^{-1}ab = a^{-1}\rangle.$$

We also need the classification of finite groups with all nontrivial elements of prime order, see e.g. [2], [3].

Theorem B. *Let G be a finite group having all nontrivial elements of prime order. Then:*
(a) *G is nilpotent if and only if G is a p-group of exponent p.*
(b) *G is solvable and non-nilpotent if and only if G is a Frobenius group with kernel $P \in \mathrm{Syl}_p(G)$ with P being a p-group of exponent p and complement $Q \in \mathrm{Syl}_q(G)$ with $|Q| = q$. Moreover, if $|G| = p^n q$ then G has a chief series $G = G_0 > P = G_1 > G_2 > \ldots > G_k > G_{k+1} = 1$ such that for every $1 \leqslant i \leqslant k$ one has $G_i/G_{i+1} \leqslant Z(P/G_{i+1})$, Q acts irreducibly on G_i/G_{i+1} and $|G_i/G_{i+1}| = p^b$, where b is the exponent of $p \pmod q$.*
(c) *G is non-solvable if and only if $G \cong A_5$.*

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [7], [12], [13]. For subgroup lattice concepts we refer the reader to [11].

2. Proofs of the main results

Proof of Theorem 1.1. Assume that G has dense normalizers. First of all, we prove that G is of square-free order. Let $|G| = p_1^{n_1} \ldots p_k^{n_k}$ and assume that $n_1 \geqslant 2$. Then G contains a subgroup P_1 of order p_1^2. By hypothesis, there is $H \leqslant G$ such that $1 < N_G(H) < P_1$. It follows that $|N_G(H)| = p_1$. Since $H \subseteq N_G(H)$, we have either $H = 1$ implying that $G = N_G(H)$ is of order p_1 — a contradiction, or $H = N_G(H)$ implying that $P_1 \subseteq H$ — a contradiction. Thus, $n_1 = \ldots = n_k = 1$.

Assume next that $k \geqslant 3$. Clearly, G cannot be cyclic. Then G is a nontrivial semidirect product of type $\mathbb{Z}_{p_1 \ldots p_r} \rtimes \mathbb{Z}_{p_{r+1} \ldots p_k}$. Since $r \geqslant 2$ or $k - r \geqslant 2$, G possesses a cyclic subgroup K of order $p_i p_j$ for some $i, j \in \{1, \ldots, k\}$. Similarly, we get that the interval $(1, K)$ of $L(G)$ contains no normalizer, a contradiction. Thus, $k \leqslant 2$ and G is either cyclic of prime order or non-Abelian of order equal to a product of two primes.

The converse is obvious, completing the proof. □

The following lemma will be useful for the proof of Theorem 1.2.

Lemma 2.1. *Let G be a finite group such that $|L(G) \setminus N_G| \leqslant 3$. Then G is of one of the following types:*

(a) *A cyclic group $\mathbb{Z}_{p^a}$ with a prime p and $a \leqslant 3$, or $\mathbb{Z}_{pq}$ with distinct primes p, q.*

(b) *A ZM-group $\mathrm{ZM}(m, n, r)$ with $\tau(m) + \tau(n) \leqslant 5$.*

Proof. First of all, we observe that G cannot have subgroups of type $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime p. Indeed, if H would be such a subgroup of G, then all $p+2$ proper subgroups of H are not normalizers and $p+2 \geqslant 4$, contradicting the hypothesis. Thus, all Abelian p-subgroups of G are cyclic. By Theorem A, it follows that Sylow subgroups of G are either cyclic or generalized quaternion 2-groups. If G has a Sylow 2-subgroup of type Q_{2^n}, then there is $Q \leqslant G$ such that $Q \cong Q_8$. We easily get that all 5 proper subgroups of Q are not normalizers, a contradiction. Consequently, the Sylow subgroups of G are cyclic, implying that G is cyclic or a ZM-group.

If $G \cong \mathbb{Z}_n$, then the condition $|L(G) \setminus N_G| \leqslant 3$ means $\tau(n) \leqslant 4$ and we get $n = p^a$ with a prime p and $a \leqslant 3$, or $n = pq$ with distinct primes p, q. If $G \cong \mathrm{ZM}(m, n, r)$, then G has a normal cyclic subgroup G_1 of order m and a cyclic subgroup G_2 of order n. Since all subgroups of G_1, as well as all proper subgroups of G_2, are not normalizers, we infer that

$$\tau(m) + \tau(n) - 2 \leqslant |L(G) \setminus N_G| \leqslant 3$$

and so $\tau(m) + \tau(n) \leqslant 5$, as desired. □

We are now able to prove Theorem 1.2.

Proof of Theorem 1.2. Items (a)–(c) follow immediately from Lemma 2.1. For item (d), assume that $|N_G| = |L(G)| - 4$ but G is not a ZM-group. Then G is either cyclic, say $G \cong \mathbb{Z}_n$, or has a noncyclic Sylow p-subgroup P. In the first case we get $\tau(n) = 5$, i.e., $n = p^4$ for some prime p, while in the second case we observe that P cannot be a generalized quaternion 2-group.[1] Then Theorem A shows that P contains a subgroup H of type $\mathbb{Z}_p \times \mathbb{Z}_p$. Since all proper subgroups of H are not normalizers, we have $p+2 \leqslant 4$, i.e., $p = 2$. Also, it is clear that H is normal in G. If $H \neq P$, then there is $K \leqslant P$ with $|K| = 8$ and $H \subset K$. Since $\mathbb{Z}_2 \times \mathbb{Z}_4$ is the unique group of order 8 having only one subgroup of type $\mathbb{Z}_2 \times \mathbb{Z}_2$, we infer that $K \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

[1] As we have seen above, such a group has at least 5 subgroups which are not normalizers.

Let $L \leqslant K$ such that $|L| = 4$ and $L \neq H$. By hypothesis, L is a normalizer, say $L = N_G(M)$ with $M \leqslant G$. Then $M \subseteq L \subset K$ and since K normalizes all its subgroups we get $K \subseteq L$, a contradiction. Thus, $H = P$, that is G has a normal Sylow 2-subgroup of type $\mathbb{Z}_2 \times \mathbb{Z}_2$.

If $G = P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we are done. Assume next that $|G| = 4p_1^{n_1} \dots p_k^{n_k}$ for some odd primes $p_1, \dots, p_k$. As in the proof of Theorem 1.1, we get $n_1 = \dots = n_k = 1$. Let $i \in \{1, \dots, k\}$ and P_i be a Sylow p_i-subgroup of G. Then P_i is a normalizer, more precisely $P_i = N_G(P_i)$. This shows that P_i is not properly contained in any Abelian subgroup of G, implying that G has no element of composite order. In other words, all nontrivial elements of G have prime order and therefore G is one of the groups described in Theorem B. Since G is not a p-group and A_5 does not satisfy the hypothesis, we infer that G is a Frobenius group with kernel $\mathbb{Z}_2 \times \mathbb{Z}_2$ and complement of order p_1. It follows that $p_1 = 3$ and $G \cong A_4$, completing the proof. □

Finally, we note that ZM-groups $G \cong \mathrm{ZM}(m, n, r)$ satisfying $|N_G| = |L(G)| - 4$ can be classified according to the inequality $\tau(m) + \tau(n) \leqslant 6$. The dihedral groups D_{30} and D_{54} are examples of such groups.

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